3.3 Role of viscosity in rotational and irrotational vortices

3.3.1 Solid body rotation of an incompressible fluid

\[ u = \frac{\omega r}{2} e_\theta \]

Solid body rotation \(\Rightarrow\) strain rate = 0

\[ \Rightarrow \text{viscous stresses} = 0 \]

\[ \Rightarrow \text{Inviscid Euler equations} \]

\[
\begin{cases}
-\rho \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} \\
0 = -\frac{\partial p}{\partial z} - \rho g
\end{cases}
\]

\[ dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz = \frac{1}{4} \rho r \omega^2 dr - \rho gz dz \]

\[ p(r,z) = \frac{1}{8} \rho \omega^2 r^2 - \rho gz + C = \frac{1}{2} \rho u_\theta^2 - \rho gz + C \]

- Bernoulli function \( B = \frac{u_\theta^2}{2} + gz + p/\rho \) not constant for different streamlines (rotational flow).
- Surfaces of constant pressure = paraboloids of revolution.
- Viscous forces are required to set up the motion.
3.3.2 Irrotational vortex in an incompressible fluid

\[ \mathbf{u} = \frac{\Gamma}{2\pi r} \mathbf{e}_\theta, \quad \omega = 0 \]

\[ \sigma_{r\theta} = \mu \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) \right] = -\frac{\mu \Gamma}{\pi r^2} \neq 0 \]

But

\[ \frac{\partial \sigma_{ij}}{\partial x_j} = \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = -\mu (\nabla \times \omega)_i = 0 \]

since

\[ (\nabla \times \omega)_i = \varepsilon_{ijk} \partial_j \omega_k = \varepsilon_{ijk} \varepsilon_{klm} \partial^2_{j,\ell} u_m \]

\[ = (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) \partial^2_{j,\ell} u_m \]

\[ = \partial^2_{j,i} u_j - \partial^2_{j,j} u_i = -\partial^2_{j,j} u_i \]

In an incompressible flow of a Newtonian fluid, the net viscous force vanishes if the flow is irrotational or if the vorticity is uniform.
Inviscid Euler equations

\[
\begin{align*}
-\rho \frac{u_\theta^2}{r} & = -\frac{\partial p}{\partial r} \\
0 & = -\frac{\partial p}{\partial z} - \rho g \\
0 & = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz
\end{align*}
\]

\[
p(r, z) = -\frac{\rho \Gamma^2}{8\pi^2 r^2} - \rho gz + C = -\frac{1}{2} \rho u_\theta^2 - \rho gz + C
\]

- Bernoulli function \( B = \frac{u_\theta^2}{2} + gz + \frac{p}{\rho} \) is a true constant (irrotational flow).
- Surfaces of constant pressure = hyperboloids of revolution.
- Viscous forces are required to maintain the motion but the net viscous force at any point is zero.
- An irrotational vortex can be created by rotating a solid circular cylinder in an infinite viscous fluid: the viscous dissipation is compensated by the work done at the surface of the cylinder.
3.4 Law of Biot and Savart.

- $\mathbf{\omega} = \nabla \times \mathbf{u} \implies \nabla \cdot \mathbf{\omega} = 0$

  Vortex lines/tubes cannot end in the fluid. They form closed loops or extend through the fluid until they hit a solid boundary.

- Vortex tubes are characterized by their (constant) total vorticity

  $$\Gamma = \int_C \mathbf{u} \cdot d\mathbf{s} = \int_{\mathcal{A}} \mathbf{\omega} \cdot \mathbf{n} \, dA$$

  Magnetic analogy: $\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$, $\nabla \cdot \mathbf{j} = 0$, $\mu_0 I = \int_C \mathbf{B} \cdot d\mathbf{s}$

  $$\mathbf{B}(s) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{j}(s') \times (s-s')}{\|s-s'\|^3} \, dV$$

  Biot Savart law:

  $$\mathbf{u}(s) = \frac{1}{4\pi} \iiint_V \frac{\mathbf{\omega}(s') \times (s-s')}{\|s-s'\|^3} \, dV$$

  For a vortex filament $C^*$ of strength $\Gamma$:

  $$\mathbf{u}(s) = \frac{\Gamma}{4\pi} \int_{C^*} \frac{ds' \times (s-s')}{{\|s-s'\|}^3}$$

  (similar to the magnetic field induced around a conductor)
Strait vortex filament = irrotational vortex

\[ \mathbf{u} = \frac{\Gamma}{2\pi r} \mathbf{e}_\theta \]

Shear layer
Interacting irrotational vortices with opposite signs

Velocity induced by the lower (upper) vortex at the center of the upper (lower) vortex:

\[ v_\star = \frac{\Gamma}{2\pi d} \]

⇒ Move forward as a pair at \( v_\star \).

Speed decreases as \( 1/r^2 \).

Interacting irrotational vortices with same sign

- Vortices circle about their midpoint.
- Speed decreases as \( 1/r \).