Internal energy budget

- $\rho \frac{D}{Dt}\left(\frac{1}{2}\|\mathbf{u}\|^2\right) = \rho \mathbf{u}\cdot\mathbf{f} + \mathbf{u}\cdot(\nabla\cdot\boldsymbol{\tau}) = \rho \mathbf{u}\cdot\mathbf{f} + \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\tau}) + p(\nabla\cdot\mathbf{u}) - \Phi$
- $\rho \frac{D}{Dt}\left(e + \frac{1}{2}\|\mathbf{u}\|^2\right) = \rho \mathbf{f}\cdot\mathbf{u} + \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\tau}) - \nabla\cdot\mathbf{q}$
- $\rho \frac{De}{Dt} = -\nabla\cdot\mathbf{q} - p(\nabla\cdot\mathbf{u}) + \Phi$

$$\int_A \mathbf{u}\cdot(\boldsymbol{\tau}\cdot\mathbf{n})\,dA = \int_A (\mathbf{u}\cdot\boldsymbol{\tau})\cdot\mathbf{n}dA = \int_V \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\tau})dV$$

- $\nabla\cdot(\mathbf{u}\cdot\boldsymbol{\tau}) = \boldsymbol{\tau} : (\nabla\mathbf{u}) + \mathbf{u}\cdot(\nabla\cdot\boldsymbol{\tau})$

- $\boldsymbol{\tau} : (\nabla\mathbf{u}) = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \tau_{ij}\left(e_{ij} + \frac{1}{2}r_{ij}\right) = \tau_{ij}e_{ij}$

  $= (-p\delta_{ij} + \sigma_{ij})e_{ij} = -p(\nabla\cdot\mathbf{u}) + \Phi$

- $\Phi = \sigma_{ij}e_{ij} = \left[2\mu e_{ij} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\delta_{ij}\right]e_{ij}$

  $= 2\mu\left[e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})\delta_{ij}\right]\left[e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})\delta_{ij}\right] \geq 0$