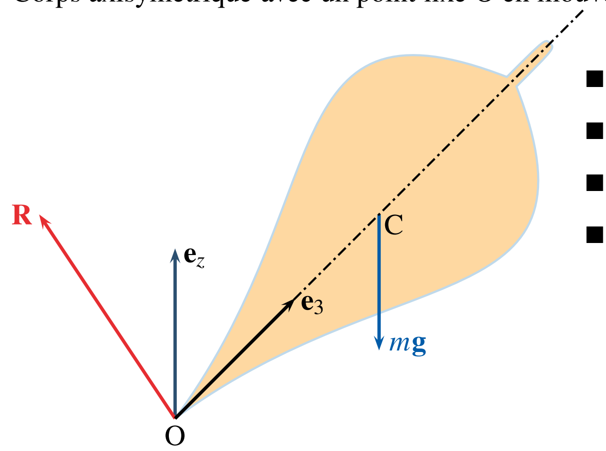
MECA0003-2 Mécanique Rationnelle

Mouvement de la toupie

Description . . . 2
Moment cinétique . . . 3
Intégrales premières . . . 4
Angles d'Euler . . . 5
Var. adim. . . . 9
Eq. adim. . . . 11
Cas général . . . 13
Précession monotone . . . 14
Préc. de signe variable . . . 15
Mvt avec rebroussement . . . 16
Précession uniforme . . . 18
Mvt asymptotique . . . 19
Toupie forte . . . 20

Description

Corps axisymétrique avec un point fixe O en mouvement dans le champ de la pesanteur.



- ddl : 6 - 3 = 3
- cg : ω
- Forces : $m\mathbf{g}$, $\mathbf{R}$
- Th. généraux :

$$\dot{\mathbf{H}}_\mathrm{O} = h\mathbf{e}_3 \wedge m\mathbf{g}$$

$$T_\mathrm{O} + V_{mg} = E$$

$\|\mathbf{OC}\| = h$

Moment cinétique

$$\dot{\mathbf{H}}_\mathrm{O} = h\mathbf{e}_3 \wedge m\mathbf{g}$$

$\cdot\, \mathbf{e}_z \quad \Rightarrow \qquad \mathbf{H}_\mathrm{O} \cdot \mathbf{e}_z = H_z = \text{constante}$

$\cdot\, \mathbf{e}_3 \quad \Rightarrow \qquad \mathbf{H}_\mathrm{O} \cdot \mathbf{e}_3 = H_3 = \text{constante}$

Corps axisymétrique : $\mathbf{J}_\mathrm{O} = A\mathbf{I} + (\Gamma - A)\mathbf{e}_3\mathbf{e}_3$

$$\mathbf{H}_\mathrm{O} = \mathbf{J}_\mathrm{O} \cdot \omega = A\omega + (\Gamma - A)\omega_3\mathbf{e}_3$$

$$\dot{\mathbf{H}}_\mathrm{O} = A\dot{\omega} + (\Gamma - A)\dot{\omega}_3\mathbf{e}_3 + (\Gamma - A)\omega_3\dot{\mathbf{e}}_3$$

$$\mathbf{e}_3 \cdot \dot{\mathbf{H}}_\mathrm{O} = A\mathbf{e}_3 \cdot \dot{\omega} + (\Gamma - A)\dot{\omega}_3 + (\Gamma - A)\omega_3\cancel{\mathbf{e}_3 \cdot \dot{\mathbf{e}}_3} = \Gamma\dot{\omega}_3$$

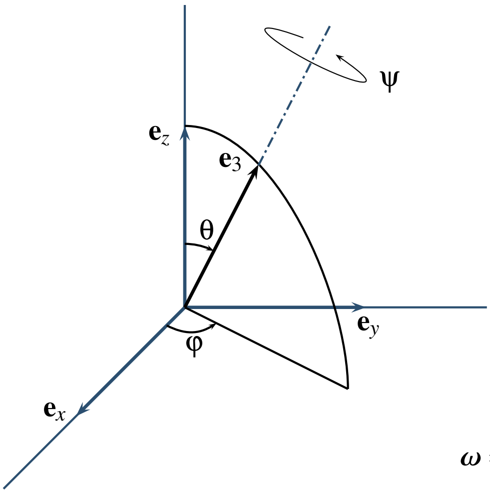
$$\Rightarrow \Gamma\omega_3 = \mathbf{H}_\mathrm{O} \cdot \mathbf{e}_3 = \Gamma n$$

Intégrales premières

$$\begin{cases} T_{\mathrm{O}} + V_{mg} = E \\ \mathbf{H}_{\mathrm{O}} \cdot \mathbf{e}_z = H_z \\ \omega_3 = n \end{cases}$$

3 intégrales premières pour trois inconnues

Angles d'Euler



- θ : angle de nutation
- φ : angle de précession
- ψ : angle de rotation propre

$$\omega = \dot{\varphi}\mathbf{e}_z + \dot{\psi}\mathbf{e}_3 + \dot{\theta}\frac{\mathbf{e}_z \wedge \mathbf{e}_3}{\|\mathbf{e}_z \wedge \mathbf{e}_3\|}$$

Intégrales premières

- $\omega_3 = n$:

$$\boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_z + \dot{\psi}\mathbf{e}_3 + \dot{\theta}\frac{\mathbf{e}_z \wedge \mathbf{e}_3}{\|\mathbf{e}_z \wedge \mathbf{e}_3\|}$$

$$\boxed{n = \boldsymbol{\omega} \cdot \mathbf{e}_3 = \dot{\varphi}\cos\theta + \dot{\psi}}$$

- $\mathbf{e}_z \cdot \mathbf{H}_\mathrm{O} = H_z$:

$$\mathbf{H}_\mathrm{O} = A\boldsymbol{\omega} + (\Gamma - A)\omega_3\mathbf{e}_3$$

$$\begin{aligned}\mathbf{e}_z \cdot \mathbf{H}_\mathrm{O} &= A\boldsymbol{\omega} \cdot \mathbf{e}_z + (\Gamma - A)n\,\mathbf{e}_3 \cdot \mathbf{e}_z \\ &= A(\dot{\varphi} + \dot{\psi}\cos\theta) + (\Gamma - A)n\cos\theta \\ &= A\left[\dot{\varphi} + \dot{\psi}\cos\theta - (\dot{\varphi}\cos\theta + \dot{\psi})\cos\theta\right] + \Gamma n\cos\theta\end{aligned}$$

$$\boxed{H_z = A\dot{\varphi}\sin^2\theta + \Gamma n\cos\theta}$$

Intégrales premières

$$\begin{aligned}T_\mathrm{O} &= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J}_\mathrm{O} \cdot \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_\mathrm{O} = \frac{1}{2}\boldsymbol{\omega} \cdot \left[A\boldsymbol{\omega} + (\Gamma - A)n\mathbf{e}_3\right] \\ &= \frac{1}{2}A\|\boldsymbol{\omega}\|^2 + \frac{1}{2}(\Gamma - A)n^2 = \frac{1}{2}A\left[\|\boldsymbol{\omega}\|^2 - n^2\right] + \frac{1}{2}\Gamma n^2 \\ &= \frac{1}{2}A\left[(\dot{\varphi}^2 + \dot{\psi}^2 + \dot{\theta}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta) - (\dot{\varphi}\cos\theta + \dot{\psi})^2\right] + \frac{1}{2}\Gamma n^2 \\ &= \frac{1}{2}A(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}\Gamma n^2\end{aligned}$$

$$V_{mg} = mgh\cos\theta$$

$$\boxed{\frac{1}{2}A\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + mgh\cos\theta = E - \frac{1}{2}\Gamma n^2}$$

Intégrales premières

$$\begin{cases} \dot\psi + \dot\varphi\cos\theta = n \\ A\dot\varphi\sin^2\theta + \Gamma n\cos\theta = H_z \\ \frac{1}{2}A\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + mgh\cos\theta = E - \frac{1}{2}\Gamma n^2 \end{cases}$$

$\left\{t,\ \theta,\ \varphi,\ \psi,\ n,\ H_z,\ E,\ A,\ \Gamma,\ m,\ g,\ h\right\}$: 12 paramètres/variables !

Passage en variables adimensionnelles

Variables adimensionnelles

$$\frac{1}{2}A\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + mgh\cos\theta = E - \frac{1}{2}\Gamma n^2$$

$$\frac{A}{2mgh}\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + \cos\theta = \frac{E - {}^1\!/_2\,\Gamma n^2}{mgh} = \alpha$$

$$\tau = \sqrt{\frac{2mgh}{A}}t \qquad \Rightarrow \qquad \frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \sqrt{\frac{2mgh}{A}}\ \mathring{(\ )}$$

$$\Rightarrow \qquad \left(\mathring{\theta}^2 + \mathring{\varphi}^2\sin^2\theta\right) + \cos\theta = \alpha$$

Variables adimensionnelles

- $n = \dot{\psi} + \dot{\varphi}\cos\theta = \sqrt{\dfrac{2mgh}{A}}\,\left(\mathring{\psi} + \mathring{\varphi}\cos\theta\right)$

$$\Rightarrow \qquad \mathring{\psi} + \mathring{\varphi}\cos\theta = n_\star = \frac{n\sqrt{A}}{\sqrt{2mgh}}$$

- $\dfrac{H_z}{\Gamma n} = \dfrac{A\dot{\varphi}\sin^2\theta + \Gamma n\cos\theta}{\Gamma n} = \dfrac{A}{\Gamma n}\sqrt{\dfrac{2mgh}{A}}\;\mathring{\varphi}\sin^2\theta + \cos\theta$

$$\beta = \frac{H_z}{\Gamma n}, \qquad b = \frac{\Gamma}{A}$$

$$\Rightarrow \qquad b n_\star(\beta - \cos\theta) = \mathring{\varphi}\sin^2\theta$$

Équations adimensionnelles

$$\begin{cases} \mathring{\psi} + \mathring{\varphi}\cos\theta = n_\star \\ b n_\star(\beta - \cos\theta) = \mathring{\varphi}\sin^2\theta \\ \mathring{\theta}^2 + \mathring{\varphi}^2\sin^2\theta + \cos\theta = \alpha \end{cases} \qquad \left\{\tau,\ \theta,\ \varphi,\ \psi,\ n_\star,\ \beta, \alpha,\ b\right\}$$

8 paramètres/variables

$$u = \cos\theta \qquad \Rightarrow \mathring{u} = -\sin\theta\,\mathring{\theta}, \quad \mathring{\theta}^2 = \frac{\mathring{u}^2}{\sin^2\theta} = \frac{\mathring{u}^2}{1-u^2}$$

$$\begin{cases} \mathring{\psi} + \mathring{\varphi}u = n_\star \\ b n_\star(\beta - u) = \mathring{\varphi}(1-u^2) \\ \dfrac{\mathring{u}^2}{1-u^2} + \mathring{\varphi}^2(1-u^2) + u = \alpha \end{cases}$$

Équations adimensionnelles

$$\begin{cases} \mathring{\psi} + \mathring{\varphi} u = n_\star & \Rightarrow \quad \mathring{\psi} = n_\star - \mathring{\varphi} u \\ b n_\star(\beta - u) = \mathring{\varphi}(1-u^2) & \Rightarrow \quad \mathring{\varphi} = \dfrac{b n_\star(\beta-u)}{1-u^2} \\ \dfrac{\mathring{u}^2}{1-u^2} + \mathring{\varphi}^2(1-u^2) + u = \alpha & \end{cases}$$

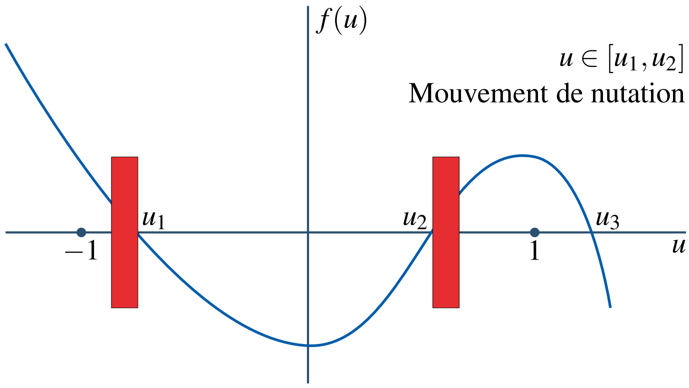
$$\mathring{u}^2 + \mathring{\varphi}^2(1-u^2)^2 = \mathring{u}^2 + b^2 n_\star^2(\beta-u)^2 = (\alpha-u)(1-u^2)$$

$$\boxed{\begin{array}{c} \mathring{u}^2 + f(u) = 0 \\ f(u) = -(\alpha-u)(1-u^2) + b^2 n_\star^2(\beta-u)^2 \end{array}}$$

$$\alpha = \frac{2E - \Gamma n^2}{2mgh}, \quad \beta = \frac{H_z}{\Gamma n}, \quad b = \frac{\Gamma}{A}, \quad n_\star = \frac{n\sqrt{A}}{\sqrt{2mgh}}$$

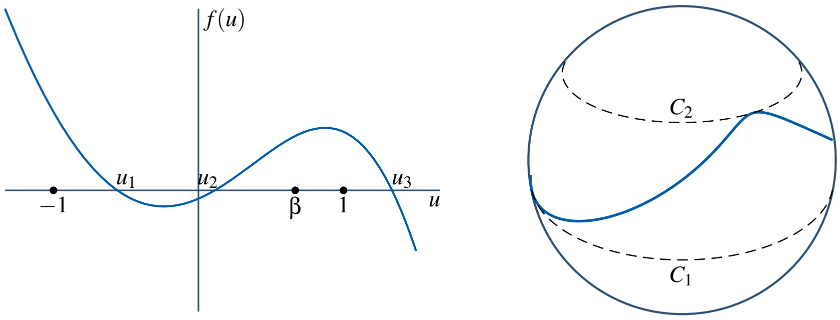
Cas général

$$\mathring{u}^2 + f(u) = 0, \qquad f(u) = -(\alpha-u)(1-u^2) + b^2 n_\star^2(\beta-u)^2$$



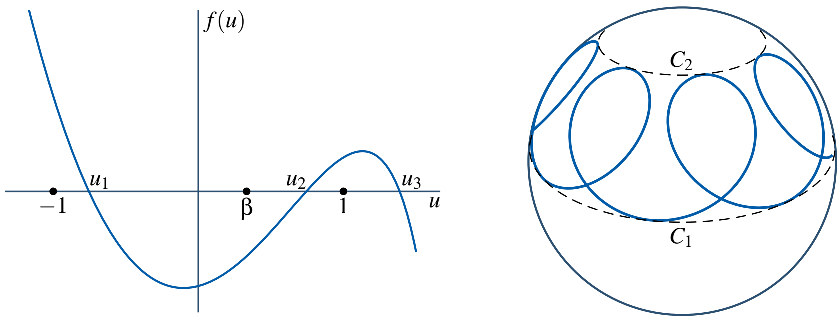
Signe de $\mathring{\varphi} = \dfrac{b n_\star(\beta-u)}{1-u^2}$ selon la position de β

Précession monotone



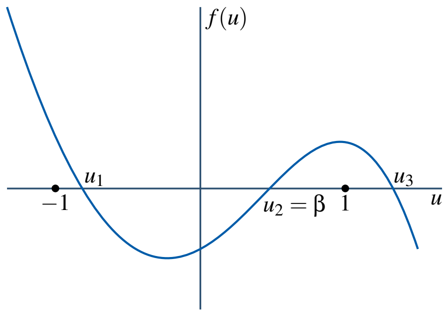
$$\mathring{\varphi} = \frac{bn_\star(\beta - u)}{1 - u^2} > 0 \quad \text{garde un signe constant.}$$

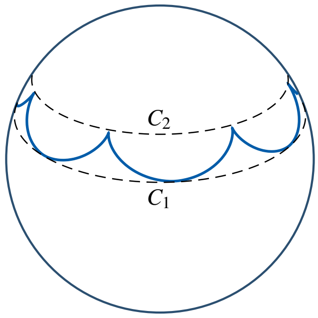
Précession de signe variable



$$\mathring{\varphi} = \frac{bn_\star(\beta - u)}{1 - u^2} \quad \text{change périodiquement de signe.}$$

Mouvement avec rebroussement





$$\frac{d\theta}{d\varphi} = \frac{\mathring{\theta}}{\mathring{\varphi}} = \mp \frac{\sqrt{-f(u)}\sqrt{1-u^2}}{bn_\star(\beta - u)} \to \infty \qquad \text{si } u \to u_2 = \beta$$

Mouvement avec rebroussement

- OK Si $\mathring{\varphi}_0 = \mathring{\theta}_0 = 0$ avec $u_0 \in]-1,1[$.

$$\mathring{\varphi}_0 = 0 = \frac{bn_\star(\beta - u_0)}{1-u_0^2} \qquad \Rightarrow \qquad \beta = u_0$$

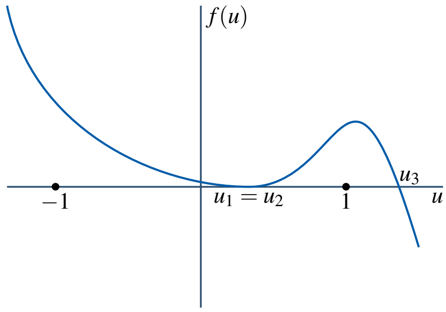
$$\mathring{u}_0^2 = 0 = -f(u_0) \qquad \Rightarrow \qquad f(\beta) = 0$$

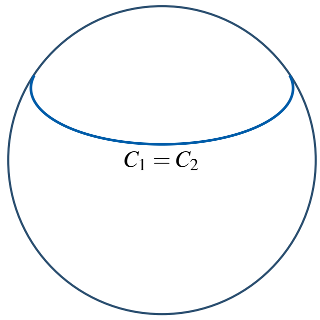
- Si $\beta \in]-1,1[$,

$$\begin{aligned} f(u) &= -(\alpha - u)(1-u) + b^2 n_\star^2 (\beta - u)^2, \quad f(\beta) = 0 \ \Rightarrow \ \alpha = \beta \\ &= -(\beta - u)(1-u) + b^2 n_\star^2 (\beta - u)^2 \end{aligned}$$

$$f'(\beta) = 1 - \beta^2 > 0 \qquad \text{Pas de rebroussement en } u_1$$

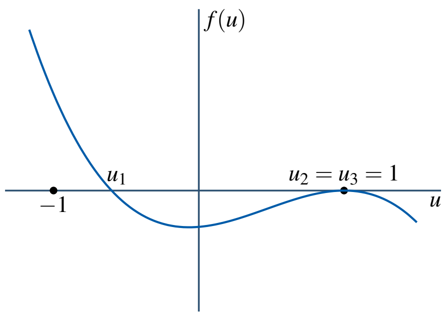
Précession uniforme

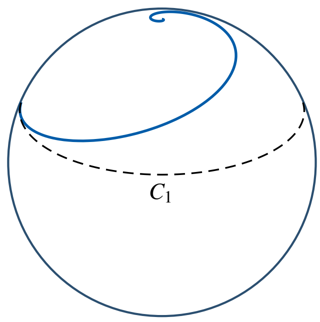




$$u = \text{constante} \qquad \Rightarrow \qquad \begin{cases} \mathring{\varphi} = \dfrac{bn_\star(\beta - u)}{1 - u^2} = \text{constante} \\[2ex] \mathring{\psi} = n_\star - \mathring{\varphi} u = \text{constante} \end{cases}$$

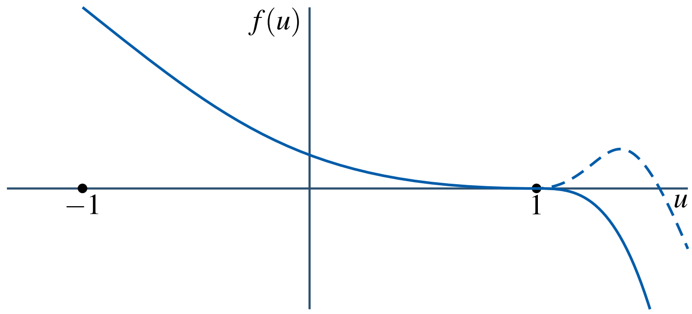
Mouvement asymptotique





Si l'axe n'est pas initialement vertical, il met un temps infini pour atteindre la verticale supérieure.

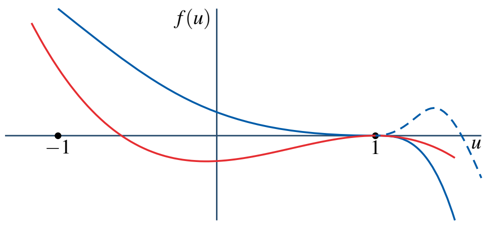
Toupie forte



La toupie est en équilibre stable en position verticale !

Toupie forte ?

$$f(u) = -(\alpha - u)(1-u^2) + b^2 n_\star^2 (\beta - u)^2$$



$f(1) = 0$

$\Rightarrow \quad \beta = 1$

$f'(1) = 0$

$\Rightarrow \quad \alpha = 1$

$$f(u) = (1-u)^2 \left[b^2 n_\star^2 - 1 - u \right]$$

$$u_3 = b^2 n_\star^2 - 1 \quad \begin{cases} < 1 & \text{Mouvement asymptotique} \\ \geq 1 & \text{Toupie forte :} \quad b^2 n_\star^2 \geq 2 \end{cases}$$